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# DIFFERENTIATION OF ENERGY FUNCTIONALS IN TWO-DIMENSIONAL ELASTICITY THEORY FOR SOLIDS WITH CURVILINEAR CRACKS

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*This paper considers the equations of two-dimensional elasticity theory in nonsmooth domains. The domains contain curvilinear cracks of variable length. On the crack faces, conditions are specified in the form of inequalities describing mutual nonpenetration of the crack faces. It is proved that the solutions of equilibrium problems with a perturbed crack converge to the solution of the equilibrium problem with an unperturbed crack in the corresponding space. The derivative of the energy functional with respect to the length of a curvilinear crack is obtained.*

**Key words:** *elasticity, crack, Griffiths criterion, variational inequality, derivative of energy functional, nonsmooth domain.*

**Introduction.** In this paper, the equilibrium problem for an elastic body is studied within the framework of two-dimensional elasticity theory. The body contains a curvilinear crack at whose faces nonpenetration conditions are specified as a system of equalities and inequalities. The body is made of a homogeneous anisotropic material which obeys Hooke's law. It is assumed that homogeneous boundary conditions are satisfied on the external boundary.

The mathematical issues of crack theory and, in particular fracture theory, which widely uses the Griffiths energy criterion, are considered. According to this criterion, crack development (propagation) begins when the derivative of the energy functional with respect to the crack length reaches the critical value of $2\gamma$, which depends on the physicomechanical properties of the material.

In the present study, a formula is obtained for the derivative of the energy functional with respect to the parameter characterizing the length of a curvilinear crack at whose faces nonpenetration conditions are specified as a system of equalities and inequalities. Strong convergence of the solution of the equilibrium problem in the perturbed domain to the solution of the equilibrium problem in the unperturbed domain is established.

The dependence of the solutions of elliptic equations on the parameters for various perturbation domains have been the subject of extensive research. The case of smooth domains is considered in [1]. Results concerning differentiation of energy functionals for linear boundary-value problems in nonsmooth domains can be found in [2, 3].

The derivative of energy functionals for nonlinear elliptic problems with conditions in the form of inequalities on the boundary was first obtained in [4]. The method of obtaining derivatives described in [4] eliminates the need for calculations of the boundary conditions for the substantial derivative of the solution, which, generally speaking, is determined ambiguously. Similar derivatives for various problems of elasticity theory were obtained in [5–10] using variational formulations [11]. The cracks were assumed to be rectilinear or additional conditions were imposed on the perturbation such that the set of admissible displacements of points of the body for the unperturbed problem is transformed in a one-to-one manner to the set of admissible displacements of points of the body for the perturbed equilibrium problem.

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The obtained formulas were used to derive Cherepanov–Rice invariant integrals [4, 5, 10]. This integral defines the rate of energy release for quasistatic crack growth and is used to describe crack extension in fracture mechanics. A mathematical basis for the invariant integral for linear problems is given in [12].

In [10], an equilibrium problem is considered for a body consisting of two homogeneous anisotropic bodies whose common boundary has a curvilinear crack with stress-free faces and a formula is obtained for the derivative of the energy functional with respect to the parameter characterizing the change in the crack length. To find the derivative of the energy functional, a coordinate transformation is used that maps the perturbed domain onto the unperturbed domain in a one-to-one manner. Because natural boundary conditions in the form of equalities are imposed on the boundary of the domain, it follows that with this coordinate transformation, the space of admissible displacements with the perturbed domain is also mapped in a one-to-one manner to the space of admissible displacements with the unperturbed domain, which is used in the derivation of the formula for the derivative. If boundary conditions with unilateral constraints are specified on the boundary of the domain, such one-to-oneness of the sets of admissible displacements is not obtained.

**Formulation of the Problem.** We consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma$, $\overline{\Omega} = \Omega \cup \Gamma$. Let a curve $\Sigma$ divides the domain $\Omega$ into two subdomains $\Omega_1$ and $\Omega_2$, i.e., $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$, $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \overline{\Sigma}$. In this case, the boundaries of the domains $\Omega_1$ and $\Omega_2$ are also piecewise smooth. The curve $\Sigma$ is specified on the plane $(x_1, x_2)$ by a function $\psi \in H^3(-l_0, l_1)$ so that $\Sigma = \{x_2 = \psi(x_1),\ -l_0 < x_1 < l_1\}$ and $l_0 > 0$, $l_1 > 0$. A segment of the curve $\Sigma$ specifies a crack $\Gamma_l$ inside the domain $\Omega$:

$$\Gamma_l = \{x_2 = \psi(x_1), \quad 0 < x_1 < l\}, \qquad 0 < l < l_1.$$

Here $l$ is a parameter that characterizes the length of the projection of $\Gamma_l$ onto the $x_1$ axis.

Let the vector $\boldsymbol{\nu} = (\nu_1, \nu_2) = (-\psi_{x_1}, 1)/\sqrt{1 + \psi_{x_1}^2}$ be a normal vector to the curve $\Sigma$. We assume that the face $\Sigma^+$ corresponds to the positive direction of the normal and $\Sigma^-$ corresponds to the negative direction.

The domain bounded by $\Gamma$, $\overline{\Gamma}_l^+$, and $\overline{\Gamma}_l^-$ will be denoted by $\Omega_0$, i.e., $\Omega_0 = \Omega \backslash \overline{\Gamma}_l$. The equilibrium problem is considered in the domain $\Omega_0$ with a nonsmooth boundary $\Gamma \cup \Gamma_l^+ \cup \Gamma_l^-$.

We introduce the displacement vector $\boldsymbol{W} = (u_1, u_2)$. The body is assumed to be made of a homogeneous elastic material which obeys Hooke's law. The strain- and stress-tensor components are given by the formulas

$$\varepsilon_{ij}(\boldsymbol{W}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \qquad \sigma_{ij}(\boldsymbol{W}) = c_{ijkl}\varepsilon_{kl}(\boldsymbol{W}) \qquad (i, j = 1, 2)$$

with a symmetric and positive definite elasticity tensor $\{c_{ijkl}\}$, i.e., $c_{ijkl} = c_{jikl} = c_{klij}$, $c_{ijkl}\xi_{kl}\xi_{ij} \geqslant c_0\xi_{ij}\xi_{ij}$, $c_0 > 0$, and $\xi_{ij} = \xi_{ji}$. For simplicity, we assume that $c_{ijkl}$ are constants.

We assume that on the external boundary, the following boundary conditions are satisfied:

$$\boldsymbol{W} = 0 \quad \text{on} \quad \Gamma. \tag{1}$$

Conditions (1) correspond to the clamping condition on the external boundary.

Let $\Pi(\Omega_0; \boldsymbol{W})$ be the potential energy functional of the body:

$$\Pi(\Omega_0; \boldsymbol{W}) = \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W})\varepsilon_{ij}(\mathbf{W}) - \int\limits_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{W}.$$

Here $\boldsymbol{f} = (f_1, f_2)$ is the specified vector of external forces; $\boldsymbol{f} \in [C^1(\overline{\Omega})]^2$.

Let us define the functional space in which the equilibrium problem will be studied. Let a subspace $H^{1,0}(\Omega_0)$ of Sobolev's space $H^1(\Omega_0)$ consist of functions which vanish on $\Gamma$. We denote the Cartesian product of two such subspaces by $H(\Omega_0)$: $H(\Omega_0) = H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$. Generally speaking, the functions from $H(\Omega_0)$ can take different values on the crack faces $\Gamma_l^+$ and $\Gamma_l^-$.

To prevent penetration of the crack faces into one another, we consider the following Singorini condition:

$$[\boldsymbol{W}] \cdot \boldsymbol{\nu} \geqslant 0 \quad \text{on} \quad \Gamma_l. \tag{2}$$

Here $[\boldsymbol{W}] = \boldsymbol{W}^+ - \boldsymbol{W}^-$ ($\boldsymbol{W}^+$ and $\boldsymbol{W}^-$ are the values of the function $\boldsymbol{W}$ at the positive and negative faces of the cut $\Gamma_l$, respectively). It should be noted that condition (2) is invariant with respect to the choice of the direction of the normal $\boldsymbol{\nu}$ because with a change in the direction to $-\boldsymbol{\nu}$, the sign of the jump $[\,\cdot\,]$ at the crack faces also changes.

We introduce the set of admissible displacements

$$K_0(\Omega_0) = \{\boldsymbol{W} \in H(\Omega_0) \mid [\boldsymbol{W}] \cdot \boldsymbol{\nu} \geqslant 0 \quad \text{almost everywhere on } \Gamma_l\},$$

which include conditions (1) on the external boundary $\Gamma$ and the nonpenetration condition (2) for the crack faces. The equilibrium problem for the body can be formulated as the problem of minimization of the energy functional $\Pi(\Omega_0; \boldsymbol{W})$ on the set of admissible displacements $K_0(\Omega_0)$:

$$\Pi(\Omega_0; \boldsymbol{W}_0) = \inf_{\boldsymbol{W} \in K_0(\Omega_0)} \Pi(\Omega_0; \boldsymbol{W}). \tag{3}$$

Because $\Pi(\Omega_0; \boldsymbol{W})$ is a coercive functional which is weakly semicontinuous from below, $K_0(\Omega_0)$ is a closed and convex set, and $H(\Omega_0)$ is Hilbert space, it follows that problem (3) has a unique solution $\boldsymbol{W}_0 \in K_0(\Omega_0)$. Since the functional $\Pi(\Omega_0; \boldsymbol{W})$ is convex and differentiable, this solution satisfies the variational inequality

$$\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_0) \geqslant \int\limits_{\Omega_0} \boldsymbol{f} \cdot (\boldsymbol{W} - \boldsymbol{W}_0) \qquad \forall\, \boldsymbol{W} \in K_0(\Omega_0). \tag{4}$$

The variational inequality (4) is equivalent to the minimization problem (3) [11].

We note that the solution of problem (3) in the domain $\Omega_0$ satisfies the equilibrium equations

$$-\frac{\partial \sigma_{ij}(\boldsymbol{W}_0)}{\partial x_j} = f_i \qquad (i = 1, 2) \quad \text{almost everywhere on} \quad \Omega_0, \tag{5}$$

the boundary condition (1), the nonpenetration condition (2), and the boundary conditions on the crack $\Gamma_l$

$$[\sigma_\nu(\boldsymbol{W}_0)] = 0, \quad \sigma_\nu(\boldsymbol{W}_0) \leqslant 0, \quad \boldsymbol{\sigma}_\tau(\boldsymbol{W}_0) = 0, \quad \sigma_\nu(\boldsymbol{W}_0)[\boldsymbol{W}_0] \cdot \boldsymbol{\nu} = 0, \tag{6}$$

which can given an exact meaning in the space $(H_{00}^{1/2}(\Gamma_l))^*$, where $(H_{00}^{1/2}(\Gamma_l))^*$ is a dual space of $H_{00}^{1/2}(\Gamma_l)$ [11]. The operators $\sigma_\nu(\boldsymbol{W})$ and $\boldsymbol{\sigma}_\tau(\boldsymbol{W}) = (\sigma_{\tau 1}(\boldsymbol{W}), \sigma_{\tau 2}(\boldsymbol{W}))$ denote normal stresses and the tangential component of the force vector on $\Gamma_l$, respectively, and are defined by the formulas

$$\{\sigma_{ij}(\boldsymbol{W})\nu_j\} = \sigma_\nu(\boldsymbol{W})\boldsymbol{\nu} + \boldsymbol{\sigma}_\tau(\boldsymbol{W}).$$

Next, we consider a set of domains with cracks that depends on the small parameter $\delta$. We define the set

$$\Gamma_{l+\delta} = \{x_2 = \psi(x_1),\ 0 < x_1 < l + \delta\}, \qquad 0 < l + \delta < l_1,$$

which characterizes the perturbation of the crack $\Gamma_l$ along the curve $\Sigma$. The domain with the crack $\Gamma_{l+\delta}$ is denoted by $\Omega_\delta = \Omega \setminus \overline{\Gamma}_{l+\delta}$. The potential energy functional of the body occupying the perturbed domain $\Omega_\delta$ is defined by

$$\Pi(\Omega_\delta; \boldsymbol{W}) = \frac{1}{2} \int\limits_{\Omega_\delta} \sigma_{ij}(\boldsymbol{W})\varepsilon_{ij}(\boldsymbol{W}) - \int\limits_{\Omega_\delta} \boldsymbol{f} \cdot \boldsymbol{W}.$$

The space $H(\Omega_\delta)$ is defined similarly to the space $H(\Omega_0)$. The set of admissible displacements of points of the body occupying the perturbed domain $\Omega_\delta$ is defined by the formula

$$K_\delta(\Omega_\delta) = \{\boldsymbol{W} \in H(\Omega_\delta) \mid [\boldsymbol{W}] \cdot \boldsymbol{\nu} \geqslant 0 \quad \text{on } \Gamma_{l+\delta}\}.$$

In the domain $\Omega_\delta$, we formulate an equilibrium problem as the problem of minimization of the energy functional on the set of admissible displacements $K_\delta(\Omega_\delta)$:

$$\Pi(\Omega_\delta; \boldsymbol{W}^\delta) = \inf_{\boldsymbol{W} \in K_\delta(\Omega_\delta)} \Pi(\Omega_\delta; \boldsymbol{W}), \tag{7}$$

which, in turn, is equivalent to the variational inequality

$$\int\limits_{\Omega_\delta} \sigma_{ij}(\boldsymbol{W}^\delta)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}^\delta) \geqslant \int\limits_{\Omega_\delta} \boldsymbol{f} \cdot (\boldsymbol{W} - \boldsymbol{W}^\delta) \qquad \forall\, \boldsymbol{W} \in K_\delta(\Omega_\delta)$$

and has a unique solution $\boldsymbol{W}^\delta \in K_\delta(\Omega_\delta)$ by virtue of the same reasons as for problem (3).

The main goal of the present study is to find the derivative of the energy functional with respect to the perturbation parameter of the domain $\Omega_0$ that characterizes the change in the crack length $\Gamma_l$, i.e., to calculate the limit

$$G = \lim_{\delta \to 0} \frac{\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta},$$

where $\boldsymbol{W}_0$ and $\boldsymbol{W}^\delta$ are the solutions of the equilibrium problem in the unperturbed and perturbed domains, respectively. The quantity $G$ characterizes the rate of energy release during quasistatic crack growth. According to the Griffiths criterion [13, 14], the crack begins to propagate when $G$ reaches a certain critical value of $2\gamma$ characteristic of the material of which the body is made. The quantity $\gamma$ defines the surface energy per unit free surface of the body; in our case, per unit crack length.

**Auxiliary Statements and Formulas.** Following [4, 8], we introduce the mapping of the perturbed domain $\Omega_\delta$ onto the initial domain $\Omega_0$. Let $B_\epsilon \subset \mathbb{R}^2$ be a sphere of radius $\epsilon > 0$ with center at the crack tip $[l, \psi(l)]$. We assume that $\epsilon$ is sufficiently small so that $\overline{B}_\epsilon \subset \Omega$ and the second tip of the crack $(0, \psi(0))$ is outside the closed sphere $\overline{B}_\epsilon$. We use a smooth cut-off function $\theta$ such that $\operatorname{supp}\theta \subset B_\epsilon$ and $\theta \equiv 1$ in $B_{\epsilon/2}$. For sufficiently small $\delta < \epsilon/2$ such that $(l+\delta, \psi(l+\delta)) \in B_\epsilon$ (this inclusion is possible because of the smoothness of the function $\psi$), we consider the transformation of the independent variables

$$y_1 = x_1 - \delta\theta(x_1, x_2), \qquad y_2 = x_2 + \psi(x_1 - \delta\theta(x_1, x_2)) - \psi(x_1) \tag{8}$$

$$((y_1, y_2) \in \Omega_0, \quad (x_1, x_2) \in \Omega_\delta),$$

which maps the perturbed domain $\Omega_\delta$ onto the unperturbed domain $\Omega_0$ in a one-to-one manner. The functional transformation matrix

$$A = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{pmatrix} 1 - \delta\theta_{,1}(\boldsymbol{x}) & (1 - \delta\theta_{,1}(\boldsymbol{x}))\psi'(x_1 - \delta\theta(\boldsymbol{x})) - \psi'(x_1) \\ -\delta\theta_{,2}(\boldsymbol{x}) & 1 - \delta\theta_{,2}(\boldsymbol{x})\psi'(x_1 - \delta\theta(\boldsymbol{x})) \end{pmatrix}$$

[$\psi'(t) = d\psi(t)/dt$ and $\boldsymbol{x} = (x_1, x_2)$] has the Jacobian

$$J_\delta = 1 - \delta\,\frac{\partial\theta}{\partial\boldsymbol{\tau}}, \qquad \frac{\partial}{\partial\boldsymbol{\tau}} \equiv \frac{\partial}{\partial x_1} + \psi'(x_1)\frac{\partial}{\partial x_2},$$

which is strictly positive for small $\delta$. The derivative $\partial/\partial\boldsymbol{\tau}$ denotes differentiation along the curve $\Sigma$, where $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ is the tangent vector to $\Sigma$.

Since the space $H^3(-l_0, l_1)$ is embedded in $C^2[-l_0 + \delta_1, l_1 - \delta_1]$, where $\delta_1 > 0$ is a sufficiently small quantity [15], the following Taylor formulas are valid in the neighborhood of $B_\epsilon$:

$$\psi'(x_1 \pm \delta\theta(\boldsymbol{x})) = \psi'(x_1) \pm \delta\theta(\boldsymbol{x})\psi''(x_1) + R_\pm(\delta, \boldsymbol{x}). \tag{9}$$

Here $R_\pm = o(\pm\delta\theta(\boldsymbol{x}))$ are additional terms in Peano form [16]. Because of the smoothness of $\theta$ and $\psi$, the following convergences hold:

$$\frac{R_\pm(\delta, \boldsymbol{x})}{\delta} \to 0 \qquad \text{strongly in} \quad L_\infty(\Omega); \tag{10}$$

in addition, $R_\pm(\delta, \cdot\,) \in H^1(\Omega)$.

By virtue of (9), the functional matrix $A$ admits the representation

$$A = I - \delta \begin{pmatrix} \theta_{,1}(\boldsymbol{x}) & \theta(\boldsymbol{x})\psi''(x_1) + \theta_{,1}(\boldsymbol{x})\psi'(x_1) \\ \theta_{,2}(\boldsymbol{x}) & \theta_{,2}(\boldsymbol{x})\psi'(x_1) \end{pmatrix} + \begin{pmatrix} 0 & R_1(\delta, \boldsymbol{x}) \\ 0 & R_2(\delta, \boldsymbol{x}) \end{pmatrix}, \tag{11}$$

where

$$\begin{gathered} R_1(\delta, \boldsymbol{x}) = R_-(\delta, \boldsymbol{x}) - \delta^2\theta(\boldsymbol{x})\theta_{,1}(\boldsymbol{x})\psi''(x_1) + \delta\theta_{,1}(\boldsymbol{x})R_-(\delta, \boldsymbol{x}), \\ R_2(\delta, \boldsymbol{x}) = \delta^2\theta_{,2}(\boldsymbol{x})\psi''(x_1) - \delta R_-(\delta, \boldsymbol{x}). \end{gathered} \tag{12}$$

From (10) and the presumed smoothnesses of the functions $\theta$ and $\psi$, it is obvious that the functions $R_i$ $(i = 1, 2)$ are uniformly bounded in $\delta$ and $\boldsymbol{x}$ for small $\delta$ and $R_i = o(\delta)$.

Since in the transformation of the independent variables (8), the domain $\Omega_\delta$ is mapped onto the domain $\Omega_0$ in a one-to-one manner, an inverse transformation $\boldsymbol{x} = \boldsymbol{x}(\delta, \boldsymbol{y})$ exists that maps the domain $\Omega_0$ onto the domain $\Omega_\delta$. We denote the transformed function $u(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_\delta$ by $\tilde{u}(\boldsymbol{y})$, $\boldsymbol{y} \in \Omega_0$, i.e., $\tilde{u}(\boldsymbol{y}) = \tilde{u}(x_1 - \delta\theta(\boldsymbol{x}), x_2 + \psi(x_1 - \delta\theta(\boldsymbol{x})) - \psi(x_1)) \equiv u(\boldsymbol{x})$. Using (11), the transformation formulas for the derivatives can be written as

$$\frac{\partial u}{\partial x_1} = \frac{\partial \tilde{u}}{\partial y_1} - \delta\theta_{,1}\frac{\partial \tilde{u}}{\partial y_1} - \delta(\theta\varphi)_{,1}\frac{\partial \tilde{u}}{\partial y_2} + R_1\frac{\partial \tilde{u}}{\partial y_2},$$
$$\frac{\partial u}{\partial x_2} = \frac{\partial \tilde{u}}{\partial y_2} - \delta\theta_{,2}\frac{\partial \tilde{u}}{\partial y_1} - \delta(\theta\varphi)_{,2}\frac{\partial \tilde{u}}{\partial y_2} + R_2\frac{\partial \tilde{u}}{\partial y_2}, \tag{13}$$

where $\varphi(x_1, x_2) = \psi'(x_1)$. Therefore, the components of the transformed strain and stress tensors become

$$\varepsilon_{ij}(\boldsymbol{W}) = \varepsilon_{ij}(\tilde{\boldsymbol{W}}) - \delta E^\delta_{ij}(\theta; \tilde{\boldsymbol{W}}) + o(\delta) r_{ij}(\tilde{\boldsymbol{W}}),$$
$$\sigma_{ij}(\boldsymbol{W}) = c_{ijkl}(\varepsilon_{kl}(\tilde{\boldsymbol{W}}) - \delta E^\delta_{kl}(\theta; \tilde{\boldsymbol{W}}) + o(\delta) r_{kl}(\tilde{\boldsymbol{W}})). \tag{14}$$

Here $r_{ij}$ are certain continuous forms that can be written in accurate form by analogy with formulas (12), using (13). In (14), we used the notation

$$E^\delta_{ij}(\theta; \boldsymbol{W}) = \frac{1}{2}\Big(\theta^\delta_{,j}\frac{\partial u_i}{\partial y_1} + \theta^\delta_{,i}\frac{\partial u_j}{\partial y_1} + (\theta\varphi)^\delta_{,j}\frac{\partial u_i}{\partial y_2} + (\theta\varphi)^\delta_{,i}\frac{\partial u_j}{\partial y_2}\Big),$$

where $\theta^\delta_{,i}(\boldsymbol{y}) = \theta_{,i}(\boldsymbol{x}(\delta, \boldsymbol{y}))$ and $(\theta\varphi)^\delta_{,i}(\boldsymbol{y}) = (\theta\varphi)_{,i}(\boldsymbol{x}(\delta, \boldsymbol{y}))$ $(i = 1, 2)$. In this case, the following convergences for $\delta \to 0$ are valid:

$$\theta^\delta_{,i} \to \theta_{,i} \qquad \text{strongly in} \quad L_\infty(\Omega_0),$$
$$(\theta\varphi)^\delta_{,i} \to (\theta\varphi)_{,i} \qquad \text{strongly in} \quad L_\infty(\Omega_0).$$

We apply the transformation of the independent variables (8) to the integrals in $\Pi(\Omega_\delta; \boldsymbol{W})$ and use formulas (14). Then, we obtain the equality $\Pi(\Omega_\delta; \boldsymbol{W}) = \Pi_\delta(\Omega_0; \tilde{\boldsymbol{W}})$ with

$$\Pi_\delta(\Omega_0; \boldsymbol{W}) = \frac{1}{2}\int\limits_{\Omega_0} J_\delta^{-1} c_{ijkl}(\varepsilon_{kl}(\boldsymbol{W}) - \delta E^\delta_{kl}(\theta; \boldsymbol{W}) + o(\delta) r_{kl}(\boldsymbol{W}))$$
$$\times (\varepsilon_{ij}(\boldsymbol{W}) - \delta E^\delta_{ij}(\theta; \boldsymbol{W}) + o(\delta) r_{ij}(\boldsymbol{W})) - \int\limits_{\Omega_0} J_\delta^{-1} \boldsymbol{f}^\delta \cdot \boldsymbol{W}, \tag{15}$$

where $\boldsymbol{f}^\delta(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}(\delta, \boldsymbol{y}))$. The set of admissible displacements $K_\delta(\Omega_\delta)$ is transformed to the set $K_\delta(\Omega_0)$ in a one-to-one manner:

$$K_\delta(\Omega_0) = \{\boldsymbol{W} \in H(\Omega_0) \mid [\boldsymbol{W}] \cdot \boldsymbol{\nu}^\delta \geqslant 0 \quad \text{on } \Gamma_l\}.$$

Here $\boldsymbol{\nu}^\delta$ is the transformed normal vector $\boldsymbol{\nu}$, i.e., $\boldsymbol{\nu}^\delta(\boldsymbol{y}) = \boldsymbol{\nu}(\boldsymbol{x}(\delta, \boldsymbol{y}))$, where $\boldsymbol{y} \in \Omega_0$ and $\boldsymbol{x} \in \Omega_\delta$. We note that the vector $\boldsymbol{\nu}^\delta$ generally does not coincide with the normal vector $\boldsymbol{\nu}$ to $\Gamma_l$. For rectilinear cracks, $\boldsymbol{\nu}^\delta = \boldsymbol{\nu} = \text{const}$ on $\Gamma_l$.

Thus, the following lemma is valid:

**Lemma 1.** *For sufficiently small* $\delta$, *the solution* $\boldsymbol{W}^\delta$ *of the perturbed problem* (7) *mapped onto the unperturbed domain* $\Omega_0$ *by means of transformation* (8) *is the unique solution* $\boldsymbol{W}_\delta \in K_\delta(\Omega_0)$ *of the problem of minimization of the functional* $\Pi_\delta(\Omega_0; \boldsymbol{W})$ *on the set* $K_\delta(\Omega_0)$. *The minimization problem is equivalent to the variational inequality*

$$\int\limits_{\Omega_0} J_\delta^{-1} c_{ijkl}(\varepsilon_{kl}(\boldsymbol{W}_\delta) - \delta E^\delta_{kl}(\theta; \boldsymbol{W}_\delta) + o(\delta) r_{kl}(\boldsymbol{W}_\delta))(\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta)$$
$$- \delta E^\delta_{ij}(\theta; \boldsymbol{W} - \boldsymbol{W}_\delta) + o(\delta) r_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta)) \geqslant \int\limits_{\Omega_0} J_\delta^{-1} \boldsymbol{f}^\delta \cdot (\boldsymbol{W} - \boldsymbol{W}_\delta), \tag{16}$$

*which is valid for all functions* $\boldsymbol{W}$ *from the set* $K_\delta(\Omega_0)$.

Substituting $\boldsymbol{W} = 0$ and $\boldsymbol{W} = 2\boldsymbol{W}_\delta$ as test functions into (16), summing the resulting inequalities, and using the Korn and Hölder inequalities, we have the uniform estimate

$$\|\boldsymbol{W}_\delta\|_{H(\Omega_0)} \leqslant c \tag{17}$$

for sufficiently small $\delta \geqslant 0$.

**Convergence of Solutions**. Because of the smoothness of the functions $\psi$ and $\boldsymbol{f}$, the operators in problem (16) can be expanded in a series in $\delta$. Indeed, we have

$$J_\delta^{-1} = 1 + \delta\,\frac{\partial\theta}{\partial\boldsymbol{\tau}} + o(\delta) \qquad \text{in} \quad \Omega_0; \tag{18}$$

$$f_i^\delta = f_i + \delta\theta\,\frac{\partial f_i}{\partial\boldsymbol{\tau}} + o(\delta) \quad \text{in} \quad \Omega_0 \qquad (i = 1, 2). \tag{19}$$

Then, relations (18) and (19) imply

$$J_\delta^{-1} f_i^\delta = f_i + \delta\,\frac{\partial}{\partial\boldsymbol{\tau}}(\theta f_i) + o(\delta) \quad \text{in} \quad \Omega_0 \qquad (i = 1, 2). \tag{20}$$

Therefore, by virtue of (20), the right side of (16) can be expanded in a series in $\delta$:

$$\int\limits_{\Omega_0} J_\delta^{-1}\boldsymbol{f}^\delta(\boldsymbol{W} - \boldsymbol{W}_\delta) = \int\limits_{\Omega_0} \Big(\Big(f_i + \delta\,\frac{\partial}{\partial\boldsymbol{\tau}}\,(\theta f_i)\Big)(u_i - u_{i\delta}) + o(\delta) r_2(\boldsymbol{W}, \boldsymbol{W}_\delta)\Big) \tag{21}$$

with a certain continuous form $r_2$.

The left side of inequality (16) can be expanded in a series in $\delta$ with a continuous form $r_3$:

$$\int\limits_{\Omega_0} J_\delta^{-1} c_{ijkl}(\varepsilon_{kl}(\boldsymbol{W}_\delta) - \delta E_{kl}^\delta(\theta; \boldsymbol{W}_\delta) + o(\delta) r_{kl}(\boldsymbol{W}_\delta))$$

$$\times\,(\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta) - \delta E_{ij}^\delta(\theta; \boldsymbol{W} - \boldsymbol{W}_\delta) + o(\delta) r_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta))$$

$$= \int\limits_{\Omega_0} \Big(\sigma_{ij}(\boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta) - \delta\Big(\sigma_{ij}(\boldsymbol{W}_\delta) E_{ij}^\delta(\theta; \boldsymbol{W} - \boldsymbol{W}_\delta)$$

$$+\, c_{ijkl} E_{kl}^\delta(\theta; \boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta) + \frac{\partial\theta}{\partial\boldsymbol{\tau}}\,\sigma_{ij}(\boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_\delta)\Big) + o(\delta) r_3(\boldsymbol{W}; \boldsymbol{W}_\delta)\Big). \tag{22}$$

To prove the theorem on the convergence of the solutions of equilibrium problems defined in perturbed domains, we need the following auxiliary lemma.

**Lemma 2.** *Let* $\boldsymbol{W}_0 = (u_{10}, u_{20}) \in K_0(\Omega_0)$ *and* $\boldsymbol{W}_\delta = (u_{1\delta}, u_{2\delta}) \in K_\delta(\Omega_0)$ *be the solutions of problems* (4) *and* (16)*, respectively. Then, the following inclusions are valid:*

$$\boldsymbol{W}_\delta^1 = \boldsymbol{W}_0 + \delta\boldsymbol{Q}_\delta^1 \in K_\delta(\Omega_0), \qquad \boldsymbol{W}_\delta^2 = \boldsymbol{W}_\delta - \delta\boldsymbol{Q}_\delta^2 \in K_0(\Omega_0).$$

*Here*

$$\boldsymbol{Q}_\delta^1 = (0, \theta^\delta\psi'' u_{10} + (R_+^\delta/\delta) u_{10}), \qquad \boldsymbol{Q}_\delta^2 = (0, \theta^\delta\psi'' u_{1\delta} + (R_+^\delta/\delta) u_{1\delta}).$$

**Proof.** Because the functions $\psi$ and $\theta$ are smooth, $\theta$ is compactly supported, and $u_{10}$ belongs to the space $H^{1,0}(\Omega_0)$, it follows that the functions $\boldsymbol{W}_\delta^1$ and $\boldsymbol{W}_\delta^2$ belong to the space $H(\Omega_0)$. Let us show that the corresponding conditions on the crack $\Gamma_l$ are also satisfied.

We consider an arbitrary function $\boldsymbol{W} \in K_\delta(\Omega_0)$ for which the following condition is satisfied:

$$[\boldsymbol{W}] \cdot \boldsymbol{\nu}^\delta \geqslant 0 \qquad \text{on} \quad \Gamma_l. \tag{23}$$

Since the coordinate transformation (8) maps the domain $\Omega_\delta$ onto the domain $\Omega_0$, we have $x_1 = y_1 + \delta\theta^\delta(\boldsymbol{y})$, where $\theta^\delta(\boldsymbol{y}) = \theta(\boldsymbol{x}(\delta, \boldsymbol{y}))$ ($\boldsymbol{y} \in \Omega_0$ and $\boldsymbol{x} \in \Omega_\delta$). By virtue of (9) and because $\boldsymbol{\nu} = (-\psi_{,1}(x_1), 1)/\sqrt{1 + \psi_{,1}^2(x_1)}$, condition (23) can be written in the following equivalent form:

$$-\psi'(y_1)[u_1] + [u_2] - \delta\theta^\delta\psi''(y_1)[u_1] - R_+^\delta(\delta, \boldsymbol{y})[u_1] \geqslant 0 \qquad \text{almost everywhere on} \quad \Gamma_l. \tag{24}$$

For the function $\boldsymbol{W}_\delta^1$ from (24), we obtain

$$-\psi'[u_{10}] + [u_{20}] + \delta\theta^\delta\psi''[u_{10}] + R_+^\delta[u_{10}] - \delta\theta^\delta\psi''[u_{10}] - R_+^\delta[u_{10}] = -\psi'[u_{10}] + [u_{20}].$$

Since $\boldsymbol{W}_0 \in K_0(\Omega_0)$, we have

$$-\psi'[u_{10}] + [u_{20}] \geqslant 0 \qquad \text{almost everywhere on} \quad \Gamma_l. \tag{25}$$

Since $\boldsymbol{W}_\delta \in K_\delta(\Omega_0)$, for the function $\boldsymbol{W}_\delta^2$ from (25) we obtain

$$\psi'[u_{1\delta}] + [u_{2\delta}] - \delta\theta^\delta\psi''[u_{1\delta}] - R_+^\delta[u_{1\delta}] \geqslant 0 \qquad \text{almost everywhere on} \quad \Gamma_l.$$

Lemma 2 is proved.

Let us prove the theorem on the convergence of the solutions of equilibrium problems defined in perturbed domains.

**Theorem 1.** *Let* $\boldsymbol{W}_0$ *be a solution of the unperturbed problem* (3)*, and let* $\boldsymbol{W}^\delta$ *be a solution of the perturbed problem* (7)*,* $\boldsymbol{W}_\delta(\boldsymbol{y}) = \boldsymbol{W}^\delta(\boldsymbol{x})$ $(\boldsymbol{y} \in \Omega_0$ *and* $\boldsymbol{x} \in \Omega_\delta)$*. Then, the following convergence for* $\delta \to 0$ *is valid*:

$$\boldsymbol{W}_\delta \to \boldsymbol{W}_0 \qquad \text{strongly in} \quad H(\Omega_0).$$

**Proof.** The functions $\boldsymbol{W}_0$ and $\boldsymbol{W}_\delta$ satisfy the variational inequalities (4) and (16), respectively. By virtue of Lemma 2, $\boldsymbol{W} = \boldsymbol{W}_\delta^2$ can be substituted as a test function into (4), and $\boldsymbol{W} = \boldsymbol{W}_\delta^1$ as a test function into (16). Using formulas (21) and (22) and summing the resulting inequalities, we have the inequality

$$\begin{aligned}
\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_\delta - \boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{W}_\delta - \boldsymbol{W}_0) &\leqslant \delta\Big(\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{Q}_\delta^1) - \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_\delta^2) \\
&+ \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_\delta)E_{ij}^\delta(\theta; \boldsymbol{W}_\delta - \boldsymbol{W}_0 - \delta\boldsymbol{Q}_\delta^1) - \int\limits_{\Omega_0} c_{ijkl}E_{kl}^\delta(\theta; \boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{W}_\delta - \boldsymbol{W}_0 - \delta\boldsymbol{Q}_\delta^1) \\
&+ \int\limits_{\Omega_0} \frac{\partial\theta}{\partial\boldsymbol{\tau}}\sigma_{ij}(\boldsymbol{W}_\delta)\varepsilon_{ij}(\boldsymbol{W}_\delta - \boldsymbol{W}_0 - \delta\boldsymbol{Q}_\delta^1) + \int\limits_{\Omega_0} \boldsymbol{f}\cdot(\boldsymbol{W}_\delta - \boldsymbol{W}_0) + \int\limits_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{Q}_\delta^2 - \int\limits_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{Q}_\delta^1 \\
&- \int\limits_{\Omega_0} \Big(\frac{\partial}{\partial\boldsymbol{\tau}}(\theta f_i)(u_{i\delta} - u_{i0} - \delta\boldsymbol{Q}_{\delta i}^1)\Big) + o(\delta)r_4(\boldsymbol{W}_0 - \delta\boldsymbol{Q}_\delta^1, \boldsymbol{W}_\delta)\Big)
\end{aligned} \tag{26}$$

with a certain bounded form $r_4$.

By virtue of the first Korn's inequality, the left side of inequality (26) is equivalent to the norm of the element $\boldsymbol{W}_\delta - \boldsymbol{W}_0$ in the space $H(\Omega_0)$. On the right side of inequality (26), the integrals at $\delta$ are bounded by virtue of (17). Thus, the following estimate uniform with respect to $\delta$ is valid:

$$\|\boldsymbol{W}_\delta - \boldsymbol{W}_0\|_{H(\Omega_0)} \leqslant c\delta.$$

Theorem 1 is proved.

Theorem 1 implies the obvious corollary.

**Corollary 1.** *The following convergences are valid:*

$$\boldsymbol{Q}_\delta^1 \to \boldsymbol{Q}_0 \qquad \textit{strongly in} \quad H(\Omega_0),$$

$$\boldsymbol{Q}_\delta^2 \to \boldsymbol{Q}_0 \qquad \textit{strongly in} \quad H(\Omega_0),$$

$$E_{ij}^\delta(\theta; \boldsymbol{W}_0) \to E_{ij}(\theta; \boldsymbol{W}_0) \qquad \textit{strongly in} \quad L_2(\Omega_0),$$

$$E_{ij}^\delta(\theta; \boldsymbol{W}_\delta) \to E_{ij}(\theta; \boldsymbol{W}_0) \qquad \textit{strongly in} \quad L_2(\Omega_0).$$

*Here*

$$\boldsymbol{Q}_0 = (0, \theta\psi''u_{10}), \qquad E_{ij}(\theta; \boldsymbol{W}) = \frac{1}{2}\Big(\theta_{,j}\frac{\partial u_i}{\partial x_1} + \theta_{,i}\frac{\partial u_j}{\partial x_1} + (\theta\varphi)_{,j}\frac{\partial u_i}{\partial x_2} + (\theta\varphi)_{,i}\frac{\partial u_j}{\partial x_2}\Big). \tag{27}$$

**Formula for the Derivative of the Energy Functional with Respect to the Crack Length.** In seeking the formula of the energy functional derivative with respect to the crack length, we use the variational properties of the solutions of equilibrium problems in the perturbed and unperturbed domains. The functional $\Pi_\delta(\Omega_0; \boldsymbol{W})$ is expanded in a series in $\delta$. Using formulas (15) and (18), we obtain

$$\Pi_\delta(\Omega_0; \boldsymbol{W}) = \frac{1}{2} \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}) \varepsilon_{ij}(\boldsymbol{W}) - \int\limits_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{W}$$

$$- \delta \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}) E^\delta_{ij}(\theta; \boldsymbol{W}) + \frac{1}{2} \delta \int\limits_{\Omega_0} \theta^\delta_{\boldsymbol{\tau}} \sigma_{ij}(\boldsymbol{W}) \varepsilon_{ij}(\boldsymbol{W}) - \delta \int\limits_{\Omega_0} \Big( \frac{\partial}{\partial \boldsymbol{\tau}} (\theta f_i) u_i + o(\delta) r_5(\boldsymbol{W}) \Big), \tag{28}$$

where $r_5$ is a certain continuous form.

We use the method proposed in [4]. By virtue of Lemma 1, the equality

$$\Pi(\Omega_\delta; \boldsymbol{W}^\delta) = \Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) \tag{29}$$

is valid for all sufficiently small $\delta > 0$. To calculate the derivative of the energy functional with respect to the crack length, it is necessary to find the limit

$$\lim_{\delta \to 0} \frac{\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta}. \tag{30}$$

Thus, by virtue of (29) and Lemma 2, we have

$$\frac{\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} = \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} \leqslant \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_0 + \delta \boldsymbol{Q}^1_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta}.$$

From this it follows that the following inequality is satisfied:

$$\limsup_{\delta \to 0} \frac{\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} \leqslant \limsup_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_0 + \delta \boldsymbol{Q}^1_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta}.$$

By virtue of the corollary of Theorem 1 and the boundedness of the form $r_5$, from formula (28) we obtain

$$\limsup_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_0 + \delta \boldsymbol{Q}^1_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} = \lim_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_0 + \delta \boldsymbol{Q}^1_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta}$$

$$= \frac{1}{2} \int\limits_{\Omega_0} \frac{\partial \theta}{\partial \boldsymbol{\tau}} \sigma_{ij}(\boldsymbol{W}_0) \varepsilon_{ij}(\boldsymbol{W}_0) - \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0) E_{ij}(\theta; \boldsymbol{W}_0) - \int\limits_{\Omega_0} \frac{\partial}{\partial \boldsymbol{\tau}} (\theta f_i) u_{i0} + \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0) \varepsilon_{ij}(\boldsymbol{Q}_0) - \int\limits_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{Q}_0.$$

At the same time, the relation

$$\frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} \geqslant \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_\delta - \delta \boldsymbol{Q}^2_\delta)}{\delta}$$

is valid, and, hence, the following inequality is satisfied:

$$\liminf_{\delta \to 0} \frac{\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0)}{\delta} \geqslant \liminf_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_\delta - \delta \boldsymbol{Q}^2_\delta)}{\delta}.$$

Taking into account Theorem 1, Corollary 1, and the boundedness of the form $r_5$ from (28), we have

$$\liminf_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_\delta - \delta \boldsymbol{Q}^2_\delta)}{\delta} = \lim_{\delta \to 0} \frac{\Pi_\delta(\Omega_0; \boldsymbol{W}_\delta) - \Pi(\Omega_0; \boldsymbol{W}_\delta - \delta \boldsymbol{Q}^2_\delta)}{\delta}$$

$$= \frac{1}{2} \int\limits_{\Omega_0} \frac{\partial \theta}{\partial \boldsymbol{\tau}} \sigma_{ij}(\boldsymbol{W}_0) \varepsilon_{ij}(\boldsymbol{W}_0) - \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0) E_{ij}(\theta; \boldsymbol{W}_0) - \int\limits_{\Omega_0} \frac{\partial}{\partial \boldsymbol{\tau}} (\theta f_i) u_{i0} + \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0) \varepsilon_{ij}(\mathbf{Q}_0) - \int\limits_{\Omega_0} \boldsymbol{f} \cdot \boldsymbol{Q}_0.$$

We find that the lower limit of the sequence $\{1/\delta(\Pi(\Omega_\delta; \boldsymbol{W}^\delta) - \Pi(\Omega_0; \boldsymbol{W}_0))\}$ is valuated from below by same constant as the upper limit of this sequence from above. Consequently, the limit (30) exists and is equal to this constant.

Thus, the following theorem is proved.

**Theorem 2.** *The derivative of the energy functional* $\Pi(\Omega_\delta; \boldsymbol{W}^\delta)$ *with respect to the length of the projection of the crack* $\Gamma_l$ *onto the* $x_1$ *axis exists and is defined by the formula*

$$\Pi'(l) = \frac{d\Pi(\Omega_\delta; \boldsymbol{W}^\delta)}{d\delta}\Big|_{\delta=0} = \frac{1}{2}\int\limits_{\Omega_0} \frac{\partial\theta}{\partial\boldsymbol{\tau}}\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{W}_0) - \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)E_{ij}(\theta; \boldsymbol{W}_0)$$
$$- \int\limits_{\Omega_0} \frac{\partial}{\partial\boldsymbol{\tau}}(\theta f_i)u_{i0} + \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0) - \int\limits_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{Q}_0, \tag{31}$$

*where* $\boldsymbol{Q}_0$ *and* $E_{ij}(\theta; \boldsymbol{W}_0)$ *are defined by formulas* (27).

**Remark 1.** Since $\Pi(\Omega_\delta; \boldsymbol{W}^\delta)$ and $\Pi(\Omega_0; \boldsymbol{W}_0)$ do not depend on the cut-off function $\theta$, the derivative $d\Pi(\Omega_\delta; \boldsymbol{W}^\delta)/d\delta\Big|_{\delta=0}$ also does not depend on $\theta$, in spite of the fact that $\theta$ appears in formula (31). This implies that for two different functions $\theta_1$ and $\theta_2$, the values of the integrals in (31) coincide.

**Remark 2.** Since formula (31) defines the derivative of the energy functional with respect to the length of the projection of the crack $\Gamma_l$ onto the $x_1$ axis, the derivative of the energy functional with respect to the length of the curvilinear crack $\Pi'(s) = \Pi'(l)((\psi'(l))^2 + 1)^{-1/2}$, where $s = \int\limits_0^l \sqrt{(\psi'(t))^2 + 1}$ is the length of the crack $\Gamma_l$.

**Analysis of the Formula.** As noted above, a formula for the derivative of the energy functional for the equilibrium problem for a body with a curvilinear crack located on the boundary of two bodies was obtained in [10]. Assuming that the elastic properties of these bodies are identical, we obtain the formulation of the problem studied in the present paper with the difference that the object of study in [10] was a crack with faces free of stresses. In this case, formula (31) differs from the corresponding formula in [10] by the last two terms, namely:

$$\Delta(\boldsymbol{W}_0) = \int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0) - \int\limits_{\Omega_0} \boldsymbol{f}\cdot\boldsymbol{Q}_0.$$

We assume that the external load $\boldsymbol{f}$ is chosen in such a manner that on the crack $\Gamma_l$ there is no contact, i.e., the crack faces are free of stresses. We show that in this case, $\Delta(\boldsymbol{W}_0) = 0$.

As is known, the following statement (generalized Green formula [11]) is valid.

**Statement 1.** *If* $\boldsymbol{U} \in H(\Omega_0)$ *and* $\sigma_{ij,j}(\boldsymbol{U}) \in L_2(\Omega_0)$, *then functionals exist that are defined on* $\Gamma_l$*:*

$$\sigma_\nu(\boldsymbol{U}), \sigma_{\tau i}(\boldsymbol{U}) \in (H_{00}^{1/2})^* \qquad (i = 1, 2);$$

*for any function* $\boldsymbol{V} = (v_1, v_2) \in [H^1(\Omega_0)]^2$ *the following formula is valid:*

$$\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{U})\varepsilon_{ij}(\boldsymbol{V}) = -\int\limits_{\Omega_0} \sigma_{ij,j}(\boldsymbol{U})v_i + \langle\sigma_\nu(\boldsymbol{U}), v_\nu\rangle_{\Gamma_l} + \langle\sigma_{\tau i}(\boldsymbol{U}), v_{\tau i}\rangle_{\Gamma_l}. \tag{32}$$

*Here* $v_\nu$ *and* $v_{\tau i}$ *are the traces of the function* $\boldsymbol{V}$ *on the crack* $\Gamma_l$ *along the normal* $\boldsymbol{\nu}$ *and the tangent* $\boldsymbol{\tau}$*, respectively. The brackets* $\langle\cdot,\cdot\rangle_{\Gamma_l}$ *denote the duality between the spaces* $H_{00}^{1/2}(\Gamma_l)$ *and* $(H_{00}^{1/2}(\Gamma_l))^*$.

Since $\boldsymbol{f} \in [C^1(\overline{\Omega})]^2 \subset [L_2(\Omega_0)]^2$, by virtue of (5), we conclude that $\sigma_{ij,j}(\boldsymbol{W}_0) \in L_2(\Omega_0)$, $i = 1, 2$. Therefore, we use formula (32) with $\boldsymbol{V} = \boldsymbol{Q}_0 \in H(\Omega_0)$. As a result, we obtain

$$\Delta(\boldsymbol{W}_0) = -\int\limits_{\Omega_0} \theta\psi'' u_{01}(\sigma_{2j,j} + f_2) + \langle\sigma_\nu(\boldsymbol{W}_0), [Q_{0\nu}]\rangle_{\Gamma_l} + \langle\sigma_{\tau i}(\boldsymbol{W}_0), [Q_{0\tau i}]\rangle_{\Gamma_l}.$$

Since the crack faces do not contact, we have $[\boldsymbol{W}_0]\cdot\boldsymbol{\nu} > 0$ on $\Gamma_l$, and, hence, by virtue of (6), we have

$$\sigma_\nu(\boldsymbol{W}_0) = 0. \tag{33}$$

Taking into account the equilibrium equations (5), the boundary conditions on the crack (6), and equality (33), we find that $\Delta(\boldsymbol{W}_0) = 0$.

In conclusion, we make a few remarks. First, if the crack $\Gamma_l$ is rectilinear, i.e., $\psi'' = 0$, formula (31) coincides with the results obtained earlier for rectilinear cracks with the nonpenetration condition for the faces [4–6].

Second, the results of the study remain valid for the case of a curvilinear crack which is a cut along a simple piecewise smooth broken curve without points of self-intersection that can be continued until intersection with the boundary of the domain $\Omega$ at a nonzero angle. In the neighborhood of the crack tip $x_1 = l$, where its perturbation occurs, the shape of the crack is defined by the equation $x_2 = \psi(x_1)$ $(x_1 \in [l - \delta_0, l + \delta_0]$ and $\delta_0 > 0)$, and the perturbation parameter $\delta \in [0, \delta_0)$. Since the behavior of the energy functional is studied for $\delta \to 0$, the parameter $\delta_0$ can be very small.

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